

A semi-classical K.A.M. theorem

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Abstract

We consider a semi-classical completely integrable system defined by a \hbar -pseudodifferential operator \hat{H} on the torus \mathbb{T}^d . In order to study perturbed operators of the form $\hat{H} + \hbar^\kappa \hat{K}$, where \hat{K} is an arbitrary pseudodifferential operator and $\kappa > 0$, we prove the conjugacy to a suitable normal form. This is then used to construct a large number of quasimodes.

1 Introduction

1.1 Semi-classical perturbations of completely integrable systems

Let us start with a short overview of the context in which this paper takes place. First of all, one knows from the Mineur-Arnol'd-Liouville Theorem [13, 2] that on a symplectic manifold \mathcal{M} , a Hamiltonian H admitting a momentum map is completely integrable (CI in short) in the sense that \mathcal{M} is fibered almost everywhere by invariant lagrangian tori on which the dynamics of H is very simple, namely conjugate to translations on the standard affine torus \mathbb{T}^d . On each torus, the trajectories are thus either periodic or quasi-periodic.

Now, one knows from Poincaré [15] that a generic perturbation $H + \varepsilon K$ will destroy its complete integrability. Nevertheless, the celebrated K.A.M. Theorem [12, 1, 14] insures that a “large part of the CI character” survives after a small perturbation εK is added. Namely, the tori on which the completely integrable dynamics satisfies a certain diophantine relation are simply slightly deformed into invariant lagrangian tori of $H + \varepsilon K$ without being destroyed by the perturbation. They are called *K.A.M. tori*.

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On the other hand, in the context of semi-classical analysis, several authors [9, 7] transposed to the pseudodifferential operators (PDOs in short) the Mineur-Arnol'd-Liouville Theorem. Thanks to these works, any PDO which admits a semi-classical momentum map can be conjugate to an operator on $T^*\mathbb{T}^d = \{(x, \xi) \mid x \in \mathbb{T}^d, \xi \in \mathbb{R}^d\}$ with a symbol depending only on ξ , microlocally in a neighborhood of any connected component of any compact regular fiber of the momentum map. We will thus work from the beginning in the angle-action coordinates and use a well-adapted pseudodifferential calculus. Actually, because of the very particular structure of the torus \mathbb{T}^d , one is able to construct a pseudodifferential calculus involving globally defined (total) symbols. These operators are sometimes called “*periodic PDOs*” in the literature. Such PDOs have been studied by several authors [8, 10, 21] but always without a small parameter. In this paper, we use a \hbar -version of these theories.

We thus begin with a PDO \hat{H} with symbol $H(\xi)$ being a CI Hamiltonian in the classical sense. It is easy to see that its spectrum is $\{H(\hbar k) \mid k \in \mathbb{Z}^d\}$ and the associated eigenvectors are simply e^{ikx} . Now, any perturbation of \hat{H} naturally relies on two parameters : a parameter ε which controls the “intensity” of the perturbation and the semi-classical parameter \hbar . In this paper, we will be interested in the regime $\varepsilon \sim \hbar^\kappa$ and thus consider perturbed operators of the form $\hat{H} + \hbar^\kappa \hat{K}$, with $\kappa > 0$ and \hat{K} any PDO.

As in the “classical” K.A.M. Theorem, one needs to impose a non-degeneracy condition on the unperturbed Hamiltonian $H(\xi)$. For example, one may require the Hessian matrix $\partial_{\xi_i} \partial_{\xi_j} H$ to be non-degenerate, i.e. Kolmogorov’s condition. We will rather use a weaker one, which will be stated precisely later on.

The perturbed operator $\hat{H} + \hbar^\kappa \hat{K}$ depends on \hbar and we want to investigate the associated family of spectra σ_\hbar depending on \hbar . Actually, the use of the pseudodifferential calculus allows to investigate \hbar^∞ -quasimodes rather than genuine eigenvectors. We recall that a quasimode is a family of functions $\varphi_\hbar \in L^2(\mathbb{T}^d)$ together with a family of numbers E_\hbar depending on \hbar , such that $\|\varphi_\hbar\|_{L^2} = 1$ and

$$\left\| \left(\hat{H} + \hbar^\kappa \hat{K} - E_\hbar \right) \varphi_\hbar \right\|_{L^2} = O(\hbar^\infty).$$

When the operator under consideration is self-adjoint, then E_\hbar is \hbar^∞ -close to the spectrum σ_\hbar ¹. The main result of this paper is the construction of a large number of quasi-modes of $\hat{H} + \hbar^\kappa \hat{K}$, as stated below².

Theorem 1. *Let \hat{H} be a PDO with non-degenerate symbol $H(\xi)$ and $\hbar^\kappa \hat{K}$ any PDO. Denote by $\langle\langle K \rangle\rangle(\xi)$ the average over the torus of the symbol $K(x, \xi)$. For*

¹Nevertheless, this does not imply that φ_\hbar is close to any eigenvector, as first remarked by Arnol'd [3].

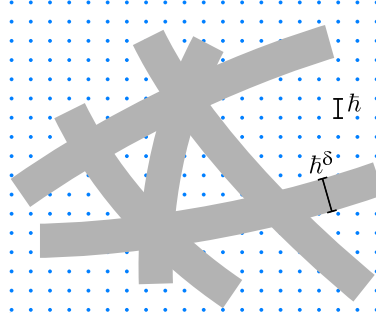
²The precise statements are given in Theorem 31 and Proposition 32.

any fixed $\delta \in (0, \frac{\kappa}{3})$, there exists a “quasi-resonant” zone $\mathcal{Z} \subset \mathbb{R}_\xi^d$ depending on \hbar and of relative volume

$$\text{vol}(\mathcal{Z}) \sim \hbar^{\delta-\varepsilon},$$

where ε can be taken arbitrarily small, and such that for all $k_\hbar \in \mathbb{Z}^d$ with $\hbar k_\hbar \in \mathbb{R}^d \setminus \mathcal{Z}$, there exists a \hbar^∞ -quasimode (φ_\hbar, E_\hbar) of $\hat{H} + \hbar^\kappa \hat{K}$ with

- $E_\hbar = H(\hbar k_\hbar) + \hbar^\kappa \langle \hat{K} \rangle(\hbar k_\hbar) + O(\hbar^{\kappa+\alpha})$, where $\alpha = \min(1 - \delta, \kappa - 3\delta)$
- $\varphi_\hbar(x) = e^{ik_\hbar x} + O(\hbar^{\kappa-1-\delta})$.



The picture above represents the case $d = 2$: the quasi-resonant zone \mathcal{Z} is in grey and the dots represent the lattice $\hbar\mathbb{Z}^d$. The assertion about the “relative volume” means that for any ball $\mathcal{O} \subset \mathbb{R}^d$ the volume of $\mathcal{Z} \cap \mathcal{O}$ is of order $\text{vol}(\mathcal{Z}) \sim \hbar^{\delta-\varepsilon}$. In the semi-classical limit, the *non-resonant set* $\mathbb{R}^d \setminus \mathcal{Z}$ tends to the set of diophantine tori which are preserved by the perturbation, due to K.A.M. theory. This should be compared to the fact that all the eigenvalues $H(\hbar k_\hbar)$ of \hat{H} with $\hbar k \in \mathbb{R}^d \setminus \mathcal{Z}$ are only slightly modified by the perturbation $\hbar^\kappa \hat{K}$ in the sense that there exists a \hbar^∞ -quasi-eigenvalue \hbar^κ -close to $H(\hbar k_\hbar)$. The first correction is the average of the symbol of the perturbation. Hence, this result should be regarded as a semi-classical K.A.M. theorem.

1.2 A result by Feldman-Knörrer-Trubowitz.

Before starting to investigate our problem, we mention the article [11] by Feldman, Knörrer and Trubowitz (FKT in short). The authors studied the high energy asymptotics for the periodic Schrödinger operator $-\Delta + V$ on the torus. Even though this problem is not in the \hbar -pseudodifferential context, one can use the “usual” correspondence *Semi-classical limit* \leftrightarrow *High frequency limit* in order to compare the results. Starting from the eigenvalue problem $-\Delta\varphi + V\varphi = \lambda\varphi$, setting $\lambda = \frac{E_\hbar}{\hbar^2}$ and multiplying everything by \hbar^2 , the problem becomes

$$-\hbar^2\Delta\varphi + \hbar^2V\varphi = E_\hbar\varphi$$

and the high energy limit $\lambda \rightarrow +\infty$ corresponds to investigating eigenvalues E_\hbar of order 1 in the semi-classical limit $\hbar \rightarrow 0$. Under this correspondence, FKT’s result appears as a special case of ours with the symbol of the

completely integrable operator $-\hbar^2\Delta$ being $H(\xi) = \xi^2$ and the perturbation being a multiplication operator of order \hbar^2 .

We now recall the relevant result of FKT. Consider the operator $-\Delta + V$ defined on the torus \mathbb{R}^d/Γ where Γ is a generic lattice of \mathbb{R}^d , V is a periodic potential and $d \leq 3$. The corresponding unperturbed operator is simply $-\Delta$ and its eigenvectors are e^{ikx} with corresponding eigenvalues k^2 , for each $k \in \Gamma^*$ and where the dual lattice Γ^* is the Fourier lattice.

Theorem 2 (FKT). *There exists an “exceptional subset” $S \subset \Gamma^*$ of density zero such that for all $k \in \Gamma^* \setminus S$ the following holds :*

- *There are 2 eigenvalues $\lambda_{\pm k}$ in the interval*

$$\left[k^2 + \langle\langle V \rangle\rangle - \frac{1}{|k|^{2-\varepsilon}}, k^2 + \langle\langle V \rangle\rangle + \frac{1}{|k|^{2-\varepsilon}} \right],$$

where $\langle\langle V \rangle\rangle$ denotes the average of V over the torus and $\varepsilon > 0$ can be taken arbitrarily small.

- *The corresponding eigenvectors $\Psi_{\pm k}$ verify*

$$\left\| \Psi_{\pm k} - \hat{\Pi} \Psi_{\pm k} \right\| = O\left(\frac{1}{|k|^{1-\varepsilon}} \right)$$

where $\hat{\Pi}$ is the projector on the span of $e^{\pm ikx}$.

The authors call *stable* the unperturbed eigenvalues k^2 for $k \notin S$ since in the large k limit, they are only slightly modified when the perturbation is added. The first correction is of order $O(1)$ and equals to the average of the “perturbation” V , and the next correction is of order $O\left(\frac{1}{|k|^{2-\varepsilon}}\right)$.

Our result (Theorem 1) extends this result to general CI PDOs, with general pseudodifferential perturbations of any order \hbar^κ and with no restriction on the dimension d . Moreover, the *exceptional subset* S arising in FKT’s result is defined in quite a tricky way. But it corresponds in our setting to the intersection of the lattice $\hbar\mathbb{Z}^d$ with the quasi-resonant zone \mathcal{Z} which is quite intuitive and geometric, as we will see in the sequel.

1.3 Normal forms and special classes of symbols

The main tool leading to Theorem 1 is a suitable semi-classical normal form for the perturbed operator $\hat{H} + \hbar^\kappa \hat{K}$, i.e. a conjugacy of $\hat{H} + \hbar^\kappa \hat{K}$ by an unitary operator $\hat{U} : \hat{U} \left(\hat{H} + \hbar^\kappa \hat{K} \right) \hat{U}^* = \widehat{NF} + O(\hbar^\infty)$, where \widehat{NF} will have special properties. The construction of this normal form is an iterative process whose first step amounts to looking for a self-adjoint PDO \hat{P} such that the conjugacy

$$e^{i\hbar^{\kappa-1}\hat{P}} \left(\hat{H} + \hbar^\kappa \hat{K} \right) e^{-i\hbar^{\kappa-1}\hat{P}} = \hat{H} + \hbar^\kappa \hat{A} + O(\hbar^{1+\kappa})$$

yields a PDO \hat{A} with the “simplest” form as possible, as we discuss just below. If we write down the corresponding equation for the symbols, we can check that the cancelation of the lower order terms is equivalent to solving

$$X_H(P) = K - A,$$

where X_H is the Hamiltonian vector field associated with the CI Hamiltonian $H(\xi)$ and is thus tangent to each torus $\mathcal{T}_\xi = \mathbb{T}^d \times \{\xi\}$. On each torus, the solutions of this *homological equation* depend strongly on the dynamics on the torus (periodic or quasiperiodic). For example, when the dynamics is periodic, then one can choose A to be the average of K along the trajectories and solve the equation with a P depending smoothly on x . On the other hand, when dynamics is quasi-periodic and satisfies a diophantine condition, then one can solve the equation, with A being the average of K over the whole torus, and obtain a P depending smoothly on x .

Unfortunately, one cannot solve this equation in that way, torus by torus, since CI Hamiltonians are generically *non-degenerate*, as assumed in Theorem 1. This property implies in particular that the vector field X_H “turns” when one moves in the space of tori. In other words, close to each periodic torus lie some quasi-periodic tori, and vice versa. It is thus impossible to solve the homological equation torus by torus since this would provide a function $P(x, \xi)$ not even continuous with respect to ξ and thus of course not acceptable as a symbol.

However, it is possible in some sense to “interpolate” between regions close to periodic motions and regions close to quasi-periodic motions, and more generally between regions close to any kind of resonance (non-resonant, partially resonant, periodic...). This can be achieved by covering the momentum space $\mathcal{B} = \{\xi\}$ with *quasi-resonant* regions, i.e. neighborhoods of resonant tori, as people usually do in Nekhoroshev-like theorems [16, 5]. In order for this construction to be of some interest, the involved neighborhoods must have a thickness which goes to 0 with \hbar , e.g. of order \hbar^δ , with $\delta > 0$. But on the other hand, this forces us to consider symbols $P(x, \xi)$ whose dependence on ξ becomes “bad” in the semi-classical limit $\hbar \rightarrow 0$. In fact, we will see that it is possible to solve the homological equation in the class Ψ_δ of symbols satisfying

$$\left| \partial_x^\alpha \partial_\xi^\beta (P_\hbar(x, \xi)) \right| \leq C_{\alpha, \beta} \hbar^{m - \delta |\beta|}.$$

These symbols are actually similar to those used by Sjöstrand in his study of the semi-excited states [20], and one can show that they indeed form an acceptable class of symbols provided $\delta < 1$.

1.4 Plan of the paper

In the next section, we give without proof the basic results concerning the pseudodifferential calculus on the torus with symbols in the class Ψ_δ . Namely, we give the composition law (Moyal's product), the L^2 -continuity (Calderón-Vaillancourt) theorem, the properties of adjoints and the functional calculus for these PDOs, and we refer to [17] for detailed proofs.

Section 3.1 is devoted to the construction of the announced covering of the momentum space $\mathcal{B} = \{\xi\}$ by quasi-resonant zones with thickness of order \hbar^δ . This construction is then used in Section 3.2 in order to define the notion of *quasi-resonant averaging* of functions. Roughly speaking, this permits to interpolate between regions with different kinds of resonance, and thus provides a function which has a different average property in each of these regions.

Equipped with these tools we are then able to study the perturbed operator $\hat{H} + \hbar^\kappa \hat{K}$ using the semi-classical normal form of Theorem 30, in which the symbol in the normal form is the quasi-resonant average of some symbol related to K . This theorem is based on Proposition 29 which insures that one can solve the homological equation, arising at each step of the normal form iteration, in the class of symbols Ψ_δ previously defined.

Finally, as an application of this normal form, we show in Theorem 31 how to build a large number of quasimodes for the perturbed operator.

2 Pseudodifferential operators on the affine torus

2.1 Classes of symbols for periodic \hbar -PDOs

We consider the cotangent bundle $T^*\mathcal{T}$ of the d -dimensional affine torus $\mathcal{T} = (\mathbb{R}/2\pi\mathbb{Z})^d$ and we will denote by (x, ξ) the canonical variables. In the following, we denote by $\Lambda^* = \mathbb{Z}^d$ the lattice of the Fourier variables, which is the 2π -dual lattice of the lattice of horizontal 1-periodic constant vector fields $\Lambda = 2\pi\mathbb{Z}^d$, i.e. vector fields of the form $X = X^j \frac{\partial}{\partial x^j}$ with $X^j \in 2\pi\mathbb{Z}$ for all (x, ξ) . We will often denote by k the Fourier variables and by $\tilde{f}(k, \xi)$ the Fourier series with respect to x of a function $f(x, \xi)$.

Definition 3. Let m and $\delta \geq 0$ be two real constants and $\mathcal{S} \subset T^*\mathcal{T}$ any subset. The **class of symbols** $\Psi_\delta^m(\mathcal{S})$ is the set of \hbar -families of functions $P_\hbar(x, \xi) \in C^\infty(T^*\mathcal{T}, \mathbb{C})$, for $\hbar \in (0, 1]$, such that for all multi-indices $\alpha, \beta \in \mathbb{Z}^d$, there exists a constant $C_{\alpha, \beta} > 0$ such that for each point $(x, \xi) \in \mathcal{S}$ and each $\hbar \in (0, 1]$, we have the following upper bound

$$\left| \partial_x^\alpha \partial_\xi^\beta (P_\hbar(x, \xi)) \right| \leq C_{\alpha, \beta} \hbar^{m - \delta|\beta|}.$$

When $\mathcal{S} = T^*\mathcal{T}$, we simply denote $\Psi_\delta^m = \Psi_\delta^m(T^*\mathcal{T})$. As well, the class of usual symbols (i.e. for $\delta = 0$) is simply denoted by $\Psi^m = \Psi_0^m(T^*\mathcal{T})$.

Moreover, it follows from the definition that $\Psi_\delta^m = \hbar^m \Psi_\delta^0$. On the other hand, the reader should keep in mind that when $\delta \neq 0$ those symbols may not have any well-defined principal symbol $\lim_{\hbar \rightarrow 0} P_\hbar$.

We denote by $\hat{\Psi}_\delta^m$ the corresponding class of PDO's, and by \hat{P} the (left) quantization of a symbol P . A PDO \hat{P} is called a **negligible operator** if $\|\hat{P}\|_{\mathcal{L}(L^2)} = O(\hbar^\infty)$. We denote this by $\hat{P} = O(\hbar^\infty)$. We say that two operators $\hat{A}, \hat{B} \in \hat{\Psi}_\delta^m$ are **equivalent** if they satisfy $\hat{A} - \hat{B} = O(\hbar^\infty)$ and we denote this by $\hat{A} \cong \hat{B}$. Note that this is slightly weaker than requiring the difference to be in $\bigcup_m \Psi_\delta^m$.

It is also convenient to have a criterion for a function to be in the class Ψ_δ^m expressed in terms of its Fourier series with respect to the x variable.

Lemma 4. *A function $P_\hbar(x, \xi)$ is a symbol in the class Ψ_δ^m if and only if its Fourier series $\tilde{P}_\hbar(k, \xi)$ satisfies the following estimate. For each multi-index $\beta \in \mathbb{Z}^d$ and each positive integer s , there exists a constant $C(s, \beta) > 0$ such that for all $k \in \Lambda^*$, all $\xi \in \mathcal{B}$ and all $\hbar \in (0, 1]$,*

$$\left| \partial_\xi^\beta \tilde{P}_\hbar(k, \xi) \right| \leq C(s, \beta) \frac{\hbar^{m-\delta|\beta|}}{(1 + |k|^2)^{\frac{s}{2}}}.$$

2.2 Composition, L^2 continuity and functional calculus

2.2.1 Asymptotic expansions

First of all, due to the presence of $\hbar^{-\delta}$ factors arising in the derivatives of symbols belonging to the class $\hat{\Psi}_\delta^m$, one is forced to consider more general asymptotic expansions than the usual ones (which read $\hbar^0 P_0(x, \xi) + \hbar^1 P_1(x, \xi) + \dots$ and which are sometimes called *classical symbols*).

Definition 5. Let $\delta \geq 0$, m and $\alpha > 0$ be real constants. Let $P_j \in \Psi_\delta^{m+j\alpha}$, $j \in \mathbb{N}$, be a sequence of symbols. We say that a symbol $P_\hbar \in \Psi_\delta^m$ admits the **asymptotic expansion**

$$P_\hbar(x, \xi) \sim \sum_{j=0}^{\infty} P_j(x, \xi, \hbar)$$

if for each integer J , one has

$$P_\hbar(x, \xi) - \sum_{j=0}^{J-1} P_j(x, \xi, \hbar) \in \Psi_\delta^{m+J\alpha}.$$

In case $\delta < 1$ and $\alpha = 1 - \delta$, P_\hbar is called a **δ -classical symbol**.

We point out that in general these asymptotic expansions are not unique since each term P_j necessarily depends on \hbar . On the other hand, one knows also that they are not convergent in general. Nevertheless, one can apply the Borel resummation process for these symbols, as stated in the following proposition (see e.g. [17].)

Lemma 6. *Let $\delta \geq 0$, m and $\alpha > 0$ be real constants. For any sequence of symbols $P_j \in \Psi_\delta^{m+j\alpha}$, there exists a symbol $P \in \Psi_\delta^m$ admitting the asymptotic expansion $P \sim \sum P_j$.*

On the other hand, a slight modification of Borel construction yields the following result.

Lemma 7. *Let m and $\alpha > 0$ be real constants. For any (non-convergent) sequence of unitary operators $\hat{U}_n \in \mathcal{L}(L^2)$ satisfying*

$$\left\| \hat{U}_n - \hat{U}_{n-1} \right\|_{\mathcal{L}(L^2)} = O(\hbar^{m+\alpha n}),$$

there exists an unitary operator \hat{U} satisfying

$$\left\| \hat{U} - \hat{U}_{n-1} \right\|_{\mathcal{L}(L^2)} = O(\hbar^{m+\alpha n}).$$

2.2.2 Composition and commutators

There is a composition law for the previously defined class of symbols Ψ_δ^m provided $\delta < 1$.

Definition 8. Let $A_\hbar, B_\hbar \in \Psi_\delta^0$. We define their **(left) Moyal product** $A_\hbar \# B_\hbar$ by

$$A_\hbar \# B_\hbar(x, \xi) = \frac{1}{(2\pi)^d} \int_{\mathcal{T}} dy \sum_{k \in \Lambda^*} e^{ik(x-y)} A_\hbar(x, \xi + \hbar k) B_\hbar(y, \xi).$$

Lemma 9. *Let $\delta \in [0, 1)$. Let \hat{A} and \hat{B} be two PDOs in the class $\hat{\Psi}_\delta^0$ with symbols A_\hbar and B_\hbar . Then, the product $\hat{C} = \hat{A}\hat{B}$ is a PDO in the same class and its symbol C_\hbar is equal to the Moyal product $C_\hbar = A_\hbar \# B_\hbar$. Moreover, the symbol C_\hbar admits the following δ -classical asymptotic expansion*

$$A_\hbar \# B_\hbar \sim \sum_{j=0}^{\infty} C_j(\hbar),$$

where the $C_j \in \Psi_\delta^{j(1-\delta)}$ are given by

$$C_j(x, \xi, \hbar) = \left(\frac{\hbar}{i}\right)^j \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha A_\hbar(x, \xi) \partial_x^\alpha B_\hbar(x, \xi).$$

From the previous lemma, one can easily obtain the symbol $A_{\hbar} \# B_{\hbar} - B_{\hbar} \# A_{\hbar}$ of the commutator $[\hat{A}, \hat{B}]$ of two PDOs. In case one of the two operators is in the class Ψ_0^0 (i.e. with $\delta = 0$) and does not depend on x , one has a slightly better expansion that will be useful in the construction of the normal form in the next section.

Lemma 10. *Let $\delta \in [0, 1)$. Let $A_{\hbar}(\xi) \in \Psi_0^0$ be a symbol independant of x and $B_{\hbar}(x, \xi) \in \Psi_{\delta}^0$ some symbol. Then the commutator $C_{\hbar} = A_{\hbar} \# B_{\hbar} - B_{\hbar} \# A_{\hbar}$ is in the class Ψ_{δ}^1 and admits an asymptotic expansion of the following form*

$$C_{\hbar}(x, \xi) \sim \frac{\hbar}{i} \{A, B\} + \sum_{j=2}^{\infty} C_j(x, \xi, \hbar),$$

where the $C_j \in \Psi_{\delta}^j$ are given by

$$C_j(x, \xi, \hbar) = \left(\frac{\hbar}{i}\right)^j \sum_{|\gamma|=j} \frac{1}{\gamma!} \partial_{\xi}^{\gamma} A_{\hbar}(\xi) \partial_x^{\gamma} B_{\hbar}(x, \xi)$$

and where the asymptotic equivalence \sim means that for each $J \in \mathbb{N}$, one has

$$C_{\hbar}(x, \xi) - \sum_{j=0}^{J-1} C_j(x, \xi, \hbar) \in \Psi_{\delta}^J.$$

2.2.3 L^2 continuity and adjoints

One can easily check that PDOs in the class $\hat{\Psi}_{\delta}^m$ are continuous from $C^{\infty}(\mathcal{T})$ to $C^{\infty}(\mathcal{T})$. Moreover, the fact that the symbols together with all its derivatives are uniformly bounded for $(x, \xi) \in T^*\mathcal{T}$ implies that the Calderón-Vaillancourt's theorem still holds in the class $\hat{\Psi}_{\delta}^0$.

Lemma 11. *Each PDO $\hat{P} \in \hat{\Psi}_{\delta}^0$ is continuous from $L^2(\mathcal{T})$ to $L^2(\mathcal{T})$. Moreover, its norm is bounded by*

$$\|\hat{P}\|_{\mathcal{L}(L^2(\mathcal{T}))} \leq C \sup_{|\gamma| \leq \frac{d+1}{2}} \sup_{x, \xi} |\partial_x^{\gamma} P_{\hbar}(x, \xi)|.$$

Let's turn now to the description of the symbol of an adjoint of a PDO.

Lemma 12. *Let $\delta \in [0, 1)$. For each $\hat{P} \in \hat{\Psi}_{\delta}^0$, the adjoint \hat{P}^* is a PDO in the same class $\hat{\Psi}_{\delta}^0$ and its symbol, denoted by P_{\hbar}^* , is given by*

$$P_{\hbar}^*(x, \xi) = \sum_{k \in \Lambda^*} \frac{1}{(2\pi)^d} \int_{\mathcal{T}} dy e^{ik(x-y)} \bar{P}_{\hbar}(y, \xi + \hbar k)$$

and admits the following δ -classical asymptotic expansion $\sum_{j=0}^{\infty} P_j^*(x, \xi, \hbar)$ where the $P_j^* \in \Psi_{\delta}^{j(1-\delta)}$ are given by

$$P_j^*(x, \xi, \hbar) = \left(\frac{\hbar}{i}\right)^j \sum_{|\gamma|=j} \frac{1}{\gamma!} \partial_x^\gamma \partial_\xi^\gamma \bar{P}_\hbar(x, \xi).$$

For convenience, we say that P_\hbar^* is the **adjoint of the symbol** P_\hbar . Moreover, a straightforward calculation shows that the Fourier series of the adjoint is given by the following expression.

Lemma 13. *Let $P_\hbar \in \Psi_{\delta}^0$ be a symbol and $P_\hbar^* \in \Psi_{\delta}^0$ its adjoint. Then their Fourier series are related as follows*

$$\widetilde{P}_\hbar^*(k, \xi) = \overline{\widetilde{P}_\hbar}(-k, \xi + \hbar k).$$

2.2.4 Exponentials and conjugacies

We want to define the exponential $e^{i\hat{P}}$ for PDOs \hat{P} in the class $\hat{\Psi}_{\delta}^m$, even for negative m . Lemma 11 insures that \hat{P} is bounded in $L^2(\mathcal{T})$, but not uniformly in \hbar , since its norm is of order \hbar^m . Nevertheless, one can define the exponential thru the resolvent formula

$$e^{i\hat{P}} = \frac{1}{2\pi i} \int_{\mathcal{C}_\hbar} e^{iz} (z - \hat{P})^{-1} dz,$$

provided $\mathcal{C}_\hbar \subset \mathbb{C}$ is a cycle surrounding the spectrum of \hat{P} , which is bounded for each \hbar . When $m < 0$, this exponential is not a PDO in the class $\hat{\Psi}_{\delta}^m$. Nevertheless, for any m it has the usual properties, namely it is unitary in $L^2(\mathcal{T})$ and satisfies $\frac{1}{i} \frac{d}{d\varepsilon} e^{i\varepsilon \hat{P}} = \hat{P} e^{i\varepsilon \hat{P}} = e^{i\varepsilon \hat{P}} \hat{P}$.

We remark in passing that for non-negative m , $e^{i\hat{P}}$ might be in the class $\hat{\Psi}_{\delta}^0$ only up to an element in $\hat{\Psi}_{\delta}^{\infty}$. The problem comes from the fact that for PDO's on the torus, the resolvent $(z - \hat{P})^{-1}$ itself may be a PDO only up to an element in $\hat{\Psi}_{\delta}^{\infty}$. We refer to [17] for a discussion of this issue.

Despite $e^{i\hat{P}}$ might not be in the class $\hat{\Psi}_{\delta}^0$, the conjugacy $\hat{C} = e^{i\hat{P}} \hat{B} e^{-i\hat{P}}$ will be, up to a negligible element, provided $m > -1$.

Lemma 14. *Let $m > -1$ and $0 \leq \delta < \min(1, 1 + m)$. Let $\hat{P} \in \hat{\Psi}_{\delta}^m$ and $\hat{B} \in \hat{\Psi}_{\delta}^0$ be two PDOs and let us consider the conjugacy $\hat{C} = e^{i\hat{P}} \hat{B} e^{-i\hat{P}}$. Then \hat{C} admits the following asymptotic expansion $\hat{C} \sim \sum_{n=0}^{\infty} \hat{C}_n$, where $\hat{C}_n \in \Psi_{\delta}^{(m+1-\delta)n}$ is given by*

$$\hat{C}_n = \frac{i^n}{n!} \underbrace{[\hat{P}, \dots, [\hat{P}, \hat{B}]]}_{n} \dots]$$

The asymptotic expansion means that, for each integer $N \geq 0$, the remainder of the truncated series verifies $\hat{C} - \sum_{n=0}^{N-1} \hat{C}_n \cong \hat{R}_N$ with $\hat{R}_N \in \hat{\Psi}_{\delta}^{(m+1-\delta)N}$.

When the operator \hat{B} is in $\hat{\Psi}_0^0$ (i.e. with $\delta = 0$) and its symbol does not depend on the x variable, then one can get a slightly better estimate (the gain is a factor \hbar^δ), that will be usefull subsequently.

Lemma 15. *Suppose now that \hat{B} is in $\hat{\Psi}_0^0$ and that its symbol $B_\hbar(\xi)$ does not depend on x . Then the PDO's \hat{C}_n of the asymptotic expansion are in $\hat{C}_n \in \Psi_\delta^{(m+1-\delta)n+\delta}$, and the remainders $\hat{R}_N \in \hat{\Psi}_\delta^{(m+1-\delta)N+\delta}$.*

3 Quasi-resonant normal form

3.1 Geometry of resonances

3.1.1 Nondegenerate Hamiltonians and resonances

Denote by $\mathcal{B} = \mathbb{R}_\xi^d$ the momentum space. A classical CI Hamiltonian $H(\xi)$ generates a linear dynamics on each torus \mathcal{T}_ξ that can be periodic, ergodic or also partially (in a sub-torus) ergodic. For each $\xi \in \mathcal{B}_\xi$, the resonant lattice of dH at the point ξ is defined by

$$\mathcal{R}_\xi = \{k \in \Lambda^*; dH_\xi(k) = 0\},$$

where Λ^* is the Fourier lattice. We thus have the following cases :

- $\dim \mathcal{R}_\xi = 0$: We say that ξ (or \mathcal{T}_ξ) is **non-resonant**. The dynamics induced by H is ergodic.
- $\dim \mathcal{R}_\xi > 0$: We say that ξ (or \mathcal{T}_ξ) is **resonant**. In this case, the dynamics is partially ergodic, i.e. ergodic in a sub-torus of dimension $d - \dim \mathcal{R}_\xi$. In particular, when $\dim \mathcal{R}_\xi = d - 1$, we say that ξ (or \mathcal{T}_ξ) is **periodic**.

From now on, the functions $dH_\xi(k)$ will be used to define the resonances and their neighborhoods.

Definition 16. For each non-vanishing $k \in \Lambda^*$, we define the fonction $\Omega_k \in C^\infty(\mathcal{B})$ by $\Omega_k(\xi) = dH_\xi(k)$ and the associated **resonance surface** $\Sigma_k \subset \mathcal{B}$ by

$$\Sigma_k = \{\xi \in \mathcal{B}; \Omega_k(\xi) = 0\}.$$

The resonant set Σ_k will indeed be a hyper-surface as soon as we will impose H to be *non-degenerate*. Such a condition is a very common assumption in K.A.M. like or Nekhoroshev like theories which insures that the CI dynamics “varies enough” from one torus to another one. The nondegeneracy condition that we will use is slightly weaker than Kolmogorov’s one [12] or Arnol’d’s one [4] and equivalent to Bryuno’s one [6]. See [18] for a review of the nondegeneracy conditions.

Definition 17. A CI Hamiltonian $H(\xi)$ is said to be **non-degenerate** if for each non-vanishing $k \in \Lambda^*$ and each point $\xi \in \Sigma_k$, we have $d(\Omega_k)_\xi \neq 0$.

This implies that the set Σ_k is a codimension 1 submanifold of \mathcal{B} and thus deserves its name “resonance surface”. Moreover, if $k_1, \dots, k_n \in \Lambda^*$ are linearly independent, then one can show (see e.g. [17]) that the submanifolds Σ_{k_j} are transverse.

3.1.2 Quasi-resonant blocks

The first step in the construction of the announced quasi-resonant normal form, is to obtain a covering of the momentum space \mathcal{B} by regions attached to each particular kind of resonance. For each resonant torus \mathcal{T}_ξ , we consider a “small” neighborhood and we remove from it a “sufficiently large” neighborhood of higher order resonances, as Pöschel did in [16], in order to get the so-called *quasi-resonant blocks*. On the other hand, in our semi-classical context, one needs to let both notions “small” and “sufficiently large” depend on \hbar . We now elaborate on Pöschel’s construction, yet incorporating \hbar in the right place. For this, we will fix two exponents $\gamma > 0$ and $\delta > 0$ which control respectively the “amount” of resonances we consider and the “size” of the quasi-resonant zones.

Definition 18. A n -dimensional sub-lattice \mathcal{R} of the Fourier lattice Λ^* is called a **resonance $\hbar^{-\gamma}$ -lattice** (or simply a **$\hbar^{-\gamma}$ -lattice**) if there exists a basis (e_1, \dots, e_n) of \mathcal{R} such that $|e_j| \leq \hbar^{-\gamma}$ for all $j = 1..n$.

Similarly with Definition 16, we define the resonant manifold attached to each $\hbar^{-\gamma}$ -lattice \mathcal{R} , making use of the function Ω_k previously defined.

Definition 19. For each resonance $\hbar^{-\gamma}$ -lattice \mathcal{R} , the associated **resonance manifold** $\Sigma_{\mathcal{R}} \subset \mathcal{B}$ is defined by

$$\Sigma_{\mathcal{R}} = \{ \xi \in \mathcal{B}; \forall k \in \mathcal{R} \Rightarrow \Omega_k(\xi) = 0 \}.$$

For consistency of the notations, in the case of the trivial lattice $\mathcal{R} = 0$, we define Σ_0 to be the whole \mathcal{B} .

For a given $\hbar^{-\gamma}$ -lattice \mathcal{R} , the resonant manifold $\Sigma_{\mathcal{R}}$ is thus the set of points ξ at which the resonant lattice of dH is exactly equal to \mathcal{R} . Moreover, we obviously have $\Sigma_{\mathcal{R}} = \bigcap_{k \in \mathcal{R}} \Sigma_k$ and the notation is still consistent when \mathcal{R} is 1-dimensional, i.e. of the form $\mathcal{R} = \mathbb{Z}.k_0$, if we write $\Sigma_{\mathbb{Z}.k_0} = \Sigma_{k_0}$. As mentioned above, the nondegeneracy hypothesis implies that for each $\hbar^{-\gamma}$ -lattice \mathcal{R} of dimension n , the manifold $\Sigma_{\mathcal{R}}$ is of codimension n in \mathcal{B} .

Definition 20. For each resonance $\hbar^{-\gamma}$ -lattice \mathcal{R} of dimension $n > 0$, the associated **resonance zone** $\mathcal{Z}_{\mathcal{R}} \subset \mathcal{B}$ is defined by

$$\mathcal{Z}_{\mathcal{R}} = \left\{ \xi \in \mathcal{B}; \forall X \in \mathbb{R}.\mathcal{R} \Rightarrow \frac{|\Omega_X(\xi)|}{|X|} < \frac{2^n \hbar^{\delta-\gamma n}}{\text{vol}(\mathcal{R})} \right\}.$$

We also define $\mathcal{Z}_0 = \mathcal{B}$.

The denominator $\text{vol}(\mathcal{R})$ in the previous definition refers to the volume of a fundamental domain of the lattice \mathcal{R} .

Definition 21. We denote by \mathcal{Z}_n^* the union of all resonance zones of order n . For $0 \leq n \leq d$, we have

$$\mathcal{Z}_n^* = \bigcup_{\dim \mathcal{R}=n} \mathcal{Z}_{\mathcal{R}}.$$

We call \mathcal{Z}_n^* the **zone of n -resonances**.

Then we remove, from each resonance zone, a neighborhood of all next order resonances and obtain the so-called resonance blocks.

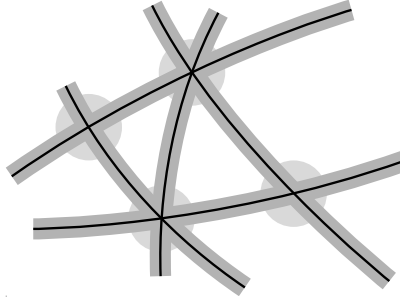
Definition 22. For each resonance $\hbar^{-\gamma}$ -lattice \mathcal{R} of dimension n , the associated **resonance block** $\mathcal{B}_{\mathcal{R}} \subset \mathcal{B}$ is defined by

$$\mathcal{B}_{\mathcal{R}} = \mathcal{Z}_{\mathcal{R}} \setminus \mathcal{Z}_{n+1}^*,$$

where we defined $\mathcal{Z}_{d+1}^* = \emptyset$ for consistency of the notations. We also denote

$$\mathcal{B}_n^* = \bigcup_{\dim \mathcal{R}=n} \mathcal{B}_{\mathcal{R}}$$

the **block of n -resonances** and call $\mathcal{B}_0^* = \mathcal{B}_0$ the **non-resonant block**.



The picture above represents the situation in dimension $d = 2$. The black lines are the resonance manifolds for 1-dimensional resonance lattices (i.e. the set of periodic tori). They intersect on resonance manifolds of 2-dimensional resonance lattices. The dark grey regions represent the associated zones of 1-resonances and the light grey regions are the zones of 2-resonances. This picture can also be understood as a 2-dimensional cross section of \mathcal{B} in dimension $d = 3$.

The resonant zones are defined in such a way (with sizes increasing with the order) that all points ξ in a given block $\mathcal{B}_{\mathcal{R}}$ are “almost resonant” for all $k \in \mathcal{R}$ and not “almost resonant” for all $k \notin \mathcal{R}$. The precise statement of this assertion is given in the following lemma. We refer the reader to Pöschel’s article ([16], p. 201) for the proof.

Lemma 23. For each resonance $\hbar^{-\gamma}$ -lattice \mathcal{R} of dimension $0 \leq n < d$ and each $\xi \in \mathcal{B}_{\mathcal{R}}$, we have

$$\forall k \notin \mathcal{R}, |k| \leq \hbar^{-\gamma} \Rightarrow \frac{|\Omega_k(\xi)|}{|k|} \geq \hbar^{\delta}.$$

This formula still holds for the non-resonant block \mathcal{B}_0 .

On the other hand, the resonance blocks form a covering of the space \mathcal{B} since they satisfy

$$\mathcal{B} = \mathcal{B}_0^* \cup \mathcal{B}_1^* \dots \cup \mathcal{B}_d^*.$$

3.2 Quasi-resonant averaging

3.2.1 Resonant averaging

For any n -dimensional resonance lattice \mathcal{R} , one can consider the averaging of functions along \mathcal{R}° , the dual space of \mathcal{R} , in the following way.

Definition 24. For any function $f \in C^\infty(T^*\mathcal{T}, \mathbb{C})$ and any n -dimensional resonance lattice \mathcal{R} , we call the **average of f with respect to \mathcal{R}** , or the **\mathcal{R} -average of f** , the function $\mathcal{R}\text{-av}(f) \in C^\infty(T^*\mathcal{T}, \mathbb{C})$ defined by

$$\mathcal{R}\text{-av}(f)(x, \xi) = \int_0^1 dt_1 \dots \int_0^1 dt_{d-n} f(x + t_1 X_1 + \dots + t_{d-n} X_{d-n}, \xi),$$

where (X_1, \dots, X_{d-n}) is any basis of $\mathcal{R}^\circ \subset \Lambda$. In particular, we will note $\langle\langle f \rangle\rangle = \{0\}\text{-av}(f)(\xi)$ the average of f over the whole torus.

One can easily check that this definition does not depend on the choice of the basis (X_1, \dots, X_{d-n}) . Moreover, it is easy to show that the Fourier series of such an averaged function has the simple form given below.

Lemma 25. Let \mathcal{R} be any resonance lattice. Let $f \in C^\infty(T^*\mathcal{T}, \mathbb{C})$ be any function and $\langle f \rangle = \mathcal{R}\text{-av}(f)$ its \mathcal{R} -average. If we denote by \widetilde{f} the Fourier series of f then $\widetilde{\langle f \rangle}$, the Fourier series of $\langle f \rangle$, is given by the following expression.

$$\widetilde{\langle f \rangle}(k, \xi) = \begin{cases} \widetilde{f}(k, \xi) & \text{for } k \in \mathcal{R} \\ 0 & \text{for } k \notin \mathcal{R}. \end{cases}$$

In particular, the Fourier series of $\langle\langle f \rangle\rangle$ verifies $\widetilde{\langle\langle f \rangle\rangle}(0, \xi) = \widetilde{f}(0, \xi)$ and vanishes for $k \neq 0$.

3.2.2 Quasi-resonant averaging

Let us consider the previously defined covering of \mathcal{B} with resonant blocks $\mathcal{B}_{\mathcal{R}}$. For any symbol $K_{\hbar} \in \Psi_{\delta}^m$, we will construct a symbol in Ψ_{δ}^m which is an \mathcal{R} -averaged function in each blocks $\mathcal{B}_{\mathcal{R}}$ and which is moreover exactly

the \mathcal{R} -average of K_h on $\Sigma_{\mathcal{R}} \cap \mathcal{B}_{\mathcal{R}}$. For the construction, we need a truncature function that will be kepted fixed and that will localize in a neighborhood of size \hbar^δ of the resonant blocks. Precisely, let us choose a function $\chi \in C_0^\infty(\mathbb{R})$ with value in $[0, 1]$, symmetric, vanishing for $|t| \geq 1$ and such that $\chi - 1$ is flat at $t = 0$.

Definition 26. For any $\delta > 0$ and any symbol $K_h \in \Psi_\delta^m$, we define A_h the \hbar^δ -average of K_h by

$$\tilde{A}_h(k, \xi) = \chi\left(\frac{\Omega_k(\xi)}{|k| \hbar^\delta}\right) \tilde{K}_h(k, \xi)$$

for all $\xi \in \mathcal{B}$ and all non-vanishing $k \in \Lambda^*$, and $\tilde{A}_h(0, \xi) = \tilde{K}_h(0, \xi)$ for all $\xi \in \mathcal{B}$.

The following lemma tells us that the \hbar^δ -average has the property that for each resonance $\hbar^{-\gamma}$ -lattice \mathcal{R} , on the resonant manifold $\Sigma_{\mathcal{R}} \cap \mathcal{B}_{\mathcal{R}}$ the \hbar^δ -average of $K_h \in \Psi_\delta^m$ is equal (up to $O(\hbar^\infty)$) to the \mathcal{R} -average of K_h , and in the resonant block $\mathcal{B}_{\mathcal{R}}$ it is a \mathcal{R} -averaged function.

Proposition 27. The \hbar^δ -average A_h of any symbol $K_h \in \Psi_\delta^m$ is in the class Ψ_δ^m and has the following properties. For each resonance $\hbar^{-\gamma}$ -lattice \mathcal{R} , we have :

$$A_h - \mathcal{R}\text{-av}(K_h) \in \Psi_\delta^\infty(\mathcal{T} \times (\Sigma_{\mathcal{R}} \cap \mathcal{B}_{\mathcal{R}})) \quad \text{and} \quad A_h - \mathcal{R}\text{-av}(A_h) \in \Psi_\delta^\infty(\mathcal{T} \times \mathcal{B}_{\mathcal{R}}).$$

Proof. Let us first show that A_h is indeed in the class Ψ_δ^m . First of all, for each multi-index $\beta \in \mathbb{N}^d$, the derivative of the Fourier series of A_h is given by

$$\partial_\xi^\beta \tilde{A}_h(k, \xi) = \sum_{\beta' \leq \beta} C_{\beta'}^\beta \left(\partial_\xi^{\beta - \beta'} \tilde{K}_h(k, \xi) \right) \partial_\xi^{\beta'} \left(\chi \left(\frac{\Omega_k(\xi)}{|k| \hbar^\delta} \right) \right)$$

for non-vanishing $k \in \Lambda^*$ and simply $\partial_\xi^\beta \tilde{A}_h(0, \xi) = \partial_\xi^\beta \tilde{K}_h(0, \xi)$ for $k = 0$. On the other hand, according to Lemma 4, the fact that K_h is in Ψ_δ^m implies for all s the estimates

$$\left| \partial_\xi^{\beta - \beta'} \tilde{K}_h(k, \xi) \right| \leq C(s, \beta - \beta') \frac{\hbar^{m - \delta |\beta - \beta'|}}{(1 + |k|^2)^{\frac{s}{2}}},$$

where $C(s, \beta - \beta')$ is a positive constant. However, one gets easily convinced that the derivatives of the function χ are of the form

$$\left| \partial_\xi^{\beta'} \left(\chi \left(\frac{\Omega_k(\xi)}{|k| \hbar^\delta} \right) \right) \right| \leq \sum_{n=1}^{|\beta'|} c(n) \hbar^{-\delta n} \chi^{(n)} \left(\frac{\Omega_k(\xi)}{|k| \hbar^\delta} \right),$$

where the constants $c(n)$ depend only on H and its derivatives. We then notice that $\hbar^{-\delta n} \leq \hbar^{-\delta|\beta'|}$ and that all the derivatives $\chi^{(n)}$ are bounded (thanks to the fact that $\chi^{(n)} \in C_0^\infty(\mathbb{R})$), what implies the estimate

$$\left| \partial_\xi^{\beta'} \left(\chi \left(\frac{\Omega_k(\xi)}{|k| \hbar^\delta} \right) \right) \right| \leq \hbar^{-\delta|\beta'|} C(\beta')$$

for all k , all \hbar and all ξ . This shows that

$$\left| \partial_\xi^\beta \tilde{A}_\hbar(k, \xi) \right| \leq C'(s, \beta) \frac{\hbar^{m-\delta|\beta-\beta'|} \hbar^{-\delta|\beta'|}}{(1+|k|^2)^{\frac{s}{2}}} = C'(s, \beta) \frac{\hbar^{m-\delta|\beta|}}{(1+|k|^2)^{\frac{s}{2}}},$$

where $C'(s, \beta)$ is a positive constant. Using again Lemma 4, we deduce that A_\hbar is a symbol in the class Ψ_δ^m .

Let us now prove that A_\hbar is an \mathcal{R} -averaged function, up to $O(\hbar^\infty)$, in each resonant block $\mathcal{B}_\mathcal{R}$. For this, let's define the remainder $R_\hbar = A_\hbar - \mathcal{R}\text{-av}(A_\hbar)$, which is in the class Ψ_δ^m since A_\hbar and $\mathcal{R}\text{-av}(A_\hbar)$ are. The Fourier series of R_\hbar is thus given by $\tilde{R}_\hbar(k, \xi) = \chi\left(\frac{\Omega_k(\xi)}{|k| \hbar^\delta}\right) \tilde{K}_\hbar(k, \xi)$ for $k \in \mathcal{R}$ and $\tilde{R}_\hbar(k, \xi) = 0$ for $k \notin \mathcal{R}$. For each $\beta \in \mathbb{N}^d$ we now estimate $\partial_\xi^\beta \tilde{R}_\hbar(k, \xi)$ at each point $\xi \in \mathcal{B}_\mathcal{R}$. For all $k \in \mathcal{R}$ with $|k| \leq \hbar^{-\gamma}$, one simply has $\partial_\xi^\beta \tilde{R}_\hbar(k, \xi) = 0$ everywhere. For all $k \notin \mathcal{R}$ with $|k| \leq \hbar^{-\gamma}$, one has $\left| \frac{\Omega_k(\xi)}{|k| \hbar^\delta} \right| \geq 1$ at each point $\xi \in \mathcal{B}_\mathcal{R}$ thanks to Lemma 23. The truncature function $\chi(t)$ vanishing for $t \geq 1$, it follows that $\partial_\xi^\beta \tilde{R}_\hbar(k, \xi) = 0$ for all $k \notin \mathcal{R}$ with $|k| \leq \hbar^{-\gamma}$ and all $\xi \in \mathcal{B}_\mathcal{R}$. Finally, for all k with $|k| > \hbar^{-\gamma}$, we simply use the fact that $R_\hbar \in \Psi_\delta^m(\mathcal{T})$, what implies (Lemma 4) that

$$\left| \partial_\xi^\beta \tilde{R}_\hbar(k, \xi) \right| \leq C(s, \beta) \frac{\hbar^{m-\delta|\beta|}}{(1+|k|^2)^{\frac{s}{2}}}, \quad (1)$$

where $C(s, \beta)$ is a positive constant.

Reconstructing the Fourier series of R_\hbar , one then gets for each $\alpha \in \mathbb{N}^d$

$$\partial_x^\alpha \partial_\xi^\beta R_\hbar(x, \xi) = \sum_{|k| > \hbar^{-\gamma}} e^{ikx} i^{|\alpha|} k^\alpha \partial_\xi^\beta \tilde{R}_\hbar(k, \xi),$$

and thus for each $\xi \in \mathcal{B}_\mathcal{R}$ one has

$$\begin{aligned} \left| \partial_x^\alpha \partial_\xi^\beta R_\hbar(x, \xi) \right| &\leq C(s, \beta) \hbar^{m-\delta|\beta|} \sum_{|k| > \hbar^{-\gamma}} \frac{|k|^{|\alpha|}}{(1+|k|^2)^{\frac{s}{2}}} \\ &\leq C(\alpha, \beta, N) \hbar^{m-\delta|\beta|} \hbar^{\gamma N}, \end{aligned}$$

where we have defined N by $s = |\alpha| + d + N$. This estimate holds for each s and thus for each N . This implies that $R_h \in \Psi_\delta^\infty(\mathcal{T} \times \mathcal{B}_\mathcal{R})$.

Let's now turn to the second property, namely on $\Sigma_\mathcal{R} \cap \mathcal{B}_\mathcal{R}$, A_h is equal up to $O(\hbar^\infty)$ to $\mathcal{R}\text{-av}(K_h)$. To prove this, we define another remainder $S_h = A_h - \mathcal{R}\text{-av}(K_h)$ whose Fourier series is given by

$$\tilde{S}_h(k, \xi) = \begin{cases} \left(\chi\left(\frac{\Omega_k(\xi)}{|k|\hbar^\delta}\right) - 1 \right) \tilde{K}_h(k, \xi) & \text{for } k \in \mathcal{R} \\ \chi\left(\frac{\Omega_k(\xi)}{|k|\hbar^\delta}\right) \tilde{K}_h(k, \xi) & \text{for } k \notin \mathcal{R} \end{cases}$$

As before, we will estimate $\partial_\xi^\beta \tilde{S}_h(k, \xi)$ for each $\beta \in \mathbb{N}^d$ at each point $\xi \in \mathcal{B}_\mathcal{R}$. For all $k \in \mathcal{R}$ with $|k| \leq \hbar^{-\gamma}$, one has $\Omega_k(\xi) = 0$ at each point $\xi \in \Sigma_\mathcal{R} \cap \mathcal{B}_\mathcal{R}$ by definition of Σ_k . On the other hand, the function $\chi(t) - 1$ is flat at $t = 0$. Thus, for all $k \in \mathcal{R}$ one has $\partial_\xi^\beta \tilde{S}_h(k, \xi) = 0$ at each $\xi \in \Sigma_\mathcal{R} \cap \mathcal{B}_\mathcal{R}$. For the cases $k \notin \mathcal{R}$ with $|k| \leq \hbar^{-\gamma}$ and $k \notin \mathcal{R}$ with $|k| > \hbar^{-\gamma}$, we argue as before and obtain that $S_h \in \Psi_\delta^\infty(\mathcal{T} \times (\Sigma_\mathcal{R} \cap \mathcal{B}_\mathcal{R}))$. \square

The \hbar^δ -average A_h of a symbol K_h has nevertheless a drawback : it is not self-adjoint even when K_h is. To solve this, we have to show that the self-adjoint part $\frac{1}{2}(A_h + A_h^*)$ has the same average properties as A_h has.

Proposition 28. *Let the parameters γ and δ satisfy $\gamma + \delta < 1$. Let $K_h \in \Psi_\delta^m$ be any self-adjoint symbol and $A_h \in \Psi_\delta^m$ its \hbar^δ -average as in Definition 26. Then the adjoint $A_h^* \in \Psi_\delta^m$ has the same properties as A_h (as given in Proposition 27), as well as its self-adjoint part $\frac{1}{2}(A_h + A_h^*)$.*

Proof. According to Lemma 13, the Fourier series of A_h^* is given by

$$\begin{aligned} \tilde{A}_h^*(k, \xi) &= \overline{\tilde{A}_h(-k, \xi + \hbar k)} \\ &= \overline{\chi\left(\frac{\Omega_{-k}(\xi + \hbar k)}{|-k|\hbar^\delta}\right)} \overline{\tilde{K}_h(-k, \xi + \hbar k)}, \end{aligned}$$

for all nonvanishing k and simply $\tilde{A}_h^*(0, \xi) = \overline{\tilde{K}_h(0, \xi)}$. Using then the facts that K_h is self-adjoint, that $\Omega_{-k} = -\Omega_k$ and that the function χ is real and symmetric, we obtain

$$\tilde{A}_h^*(k, \xi) = \chi\left(\frac{\Omega_k(\xi + \hbar k)}{|k|\hbar^\delta}\right) \tilde{K}_h(k, \xi),$$

for all non-vanishing k and $\tilde{A}_h^*(0, \xi) = \tilde{K}_h(0, \xi)$. We now prove that $A_h^* - \mathcal{R}\text{-av}(A_h^*) \in \Psi_\delta^\infty(\mathcal{T} \times \mathcal{B}_\mathcal{R})$. We introduce the remainder $R_h = A_h^* - \mathcal{R}\text{-av}(A_h^*)$, which is proved to be in the class Ψ_δ^m in the same way as we proved it for A_h itself. The Fourier series of R_h is given by $\tilde{R}_h(k, \xi) = \chi\left(\frac{\Omega_k(\xi + \hbar k)}{|k|\hbar^\delta}\right) \tilde{K}_h(k, \xi)$ for all $k \notin \mathcal{R}$ and 0 otherwise.

For each $\beta \in \mathbb{N}^d$ we have to estimate $\partial_\xi^\beta \tilde{R}_h(k, \xi)$ at each point $\xi \in \mathcal{B}_R$. For all $k \in \mathcal{R}$ with $|k| \leq \hbar^{-\gamma}$, one simply has $\partial_\xi^\beta \tilde{R}_h(k, \xi) = 0$ everywhere. For all $k \notin \mathcal{R}$ with $|k| \leq \hbar^{-\gamma}$, one has $\left| \frac{\Omega_k(\xi)}{|k|\hbar^\delta} \right| \geq 1$ at each point $\xi \in \mathcal{B}_R$ thanks to Lemma 23. Nevertheless, we have to evaluate Ω_k at $\xi + \hbar k$ and not at ξ . However, the bound $|k| \leq \hbar^{-\gamma}$ insures that $\frac{\Omega_k(\xi + \hbar k)}{|k|\hbar^\delta} = \frac{\Omega_k(\xi)}{|k|\hbar^\delta} + O(\hbar^{1-\delta-\gamma})$. The relation between δ and γ then implies that $\hbar^{1-\delta-\gamma}$ is small when $\hbar \rightarrow 0$, and using the fact that the truncature function $\chi(t)$ is flat when $|t| \geq 1$, we obtain $\chi\left(\frac{\Omega_k(\xi + \hbar k)}{|k|\hbar^\delta}\right) = O(\hbar^{(1-\delta-\gamma)\infty}) = O(\hbar^\infty)$. A similar argument for the derivatives $\chi^{(n)}$ yields $\partial_\xi^\beta \tilde{R}_h(k, \xi) = O(\hbar^{-|\beta|\delta} \hbar^{(1-\delta-\gamma)\infty}) = O(\hbar^\infty)$ for all $k \notin \mathcal{R}$ with $|k| \leq \hbar^{-\gamma}$ and all $\xi \in \mathcal{B}_R$. Finally, for all k with $|k| > \hbar^{-\gamma}$, we use as before the fact that $R_h \in \Psi_\delta^m(\mathcal{T})$, what implies (Lemma 4) that

$$\left| \partial_\xi^\beta \tilde{R}_h(k, \xi) \right| \leq C(s, \beta) \frac{\hbar^{m-\delta|\beta|}}{(1 + |k|^2)^{\frac{s}{2}}},$$

where $C(s, \beta)$ is a positive constant.

Reconstructing the Fourier series of R_h , one gets for each $\alpha \in \mathbb{N}^d$

$$\partial_x^\alpha \partial_\xi^\beta R_h(x, \xi) = \sum_{|k| \leq \hbar^{-\gamma}} e^{ikx} i^{|\alpha|} k^\alpha \partial_\xi^\beta \tilde{R}_h(k, \xi) + \sum_{|k| > \hbar^{-\gamma}} e^{ikx} i^{|\alpha|} k^\alpha \partial_\xi^\beta \tilde{R}_h(k, \xi),$$

and thus for each $\xi \in \mathcal{B}_R$ one has

$$\left| \partial_x^\alpha \partial_\xi^\beta R_h(x, \xi) \right| \leq O(\hbar^\infty) \sum_{|k| \leq \hbar^{-\gamma}} |k|^{|\alpha|} + C(s, \beta) \hbar^{m-\delta|\beta|} \sum_{|k| > \hbar^{-\gamma}} \frac{|k|^{|\alpha|}}{(1 + |k|^2)^{\frac{s}{2}}}.$$

This holds for all s and thus one has $\left| \partial_x^\alpha \partial_\xi^\beta R_h(x, \xi) \right| = O(\hbar^\infty)$. This proves that $R_h \in \Psi_\delta^\infty(\mathcal{T} \times \mathcal{B}_R)$.

We now let the reader check that, following the same arguments, one can show that $A_h^* - \mathcal{R}\text{-av}(K_h^*) \in \Psi_\delta^\infty(\mathcal{T} \times \Sigma_R)$. This shows that A_h^* has the same average properties as A_h , as well as $\frac{1}{2}(A_h + A_h^*)$. \square

From now on, the function A_h in Definition 26 is called the \hbar^δ -**average** of K_h and the one in Proposition 28 is called the **self-adjoint** \hbar^δ -**average** of K_h .

3.3 Semi-classical normal form

Let's now turn to the study of the *homological equation* arising at each step of the construction of the normal form given in Theorem 30.

Proposition 29 (Homological equation). *Let $H(\xi) \in \Psi^0$ be a non-degenerate CI Hamiltonian. Let $K_h \in \Psi_\delta^m$ be any symbol and $A_h \in \Psi_\delta^m$ its \hbar^δ -average. Then there exists a symbol $P_h \in \Psi_\delta^{m-\delta}$ solution of the equation*

$$\{P_h, H\} + K_h - A_h = 0.$$

Proof. We first write the Fourier series of the homological equation, i.e.

$$i\Omega_k(\xi) \tilde{P}_h(k, \xi) + \tilde{K}_h(k, \xi) - \tilde{A}_h(k, \xi) = 0. \quad (2)$$

For $k = 0$, the equation is fulfilled since $\Omega_0(\xi) = 0$ and $\tilde{A}_h(0, \xi) = \tilde{K}_h(0, \xi)$. We can choose for example $\tilde{P}_h(k, \xi) = 0$. For all $k \neq 0$, the Fourier series of the \hbar^δ -average is given by $\tilde{A}_h(k, \xi) = \chi\left(\frac{\Omega_k(\xi)}{|k|\hbar^\delta}\right) \tilde{K}_h(k, \xi)$. We then notice that the function $\phi(t) = \frac{1-\chi(t)}{t}$ is smooth. This implies that the function $P_h(x, \xi)$ defined by

$$\tilde{P}_h(k, \xi) = \frac{i\tilde{K}_h(k, \xi)}{|k|\hbar^\delta} \phi\left(\frac{\Omega_k(\xi)}{|k|\hbar^\delta}\right)$$

is well-defined and satisfies Equation 2. Moreover, proceeding as in Proposition 27 for proving that the \hbar^δ -average is a symbol in the class Ψ_δ^m , and using the smoothness of ϕ , one shows that for all $\alpha, \beta \in \mathbb{N}^d$,

$$\left| \partial_x^\alpha \partial_\xi^\beta P_h(x, \xi) \right| \leq C(\alpha, \beta) \hbar^{m-\delta-|\beta|}.$$

This proves that $P_h \in \Psi_\delta^{m-\delta}$. \square

Theorem 30 (Semi-classical normal form). *Let us consider a PDO $\hat{H} \in \hat{\Psi}^0$ with non-degenerate CI symbol $H(\xi) \in \Psi^0$ and any self-adjoint perturbation $\hbar^\kappa \hat{K}_0 \in \hat{\Psi}_\delta^\kappa$, with $\kappa > 0$. Let us choose small parameters $\gamma \geq 0$ and $\delta \geq 0$ such that $\delta < 1 - \gamma$ and $\delta < \frac{\kappa}{3}$, and let us consider the covering of \mathcal{B} with resonance blocks as described in Section 3.1.2.*

Then there exist an unitary operator \hat{U} and a self-adjoint PDO $\hat{K} \in \hat{\Psi}_\delta^0$ satisfying

$$\hat{U} \left(\hat{H} + \hbar^\kappa \hat{K}_0 \right) \hat{U}^{-1} = \hat{H} + \hbar^\kappa \hat{A} + O(\hbar^\infty), \quad (3)$$

where the symbol of $\hat{A} \in \hat{\Psi}_\delta^0$ is the self-adjoint \hbar^δ -average of the symbol K_h . Moreover, one has $\hat{U} = \mathbb{I} + O(\hbar^{\kappa-1-\delta})$ and $\hat{K} = \hat{K}_0 + O(\hbar^\alpha)$, with $\alpha = \min(1 - \delta, \kappa - 3\delta)$.

Proof. The exponent α is positive because of the restrictions on δ and κ . We first prove that there exist self-adjoint PDOs $\hat{P}_0 \in \hat{\Psi}_\delta^{-\delta}(\mathcal{T})$ and $\hat{K}_1 \in \hat{\Psi}_\delta^\alpha(\mathcal{T})$ such that

$$e^{i\hbar^{\kappa-1}\hat{P}_0} \left(\hat{H} + \hbar^\kappa \hat{K}_0 \right) e^{-i\hbar^{\kappa-1}\hat{P}_0} \cong \hat{H} + \hbar^\kappa \hat{A}_0 + \hbar^\kappa \hat{K}_1, \quad (4)$$

where $A_0(\hbar) \in \Psi_\delta^0$ is the self-adjoint \hbar^δ -average of $K_0(\hbar)$.

• Indeed, Lemma 15 tells us that

$$\begin{aligned} e^{i\hbar^{\kappa-1}\hat{P}_0}\hat{H}e^{-i\hbar^{\kappa-1}\hat{P}_0} &\cong \hat{H} + i\hbar^{\kappa-1}[\hat{P}_0, \hat{H}] + O\left(\hbar^{2(\kappa-1-\delta+1-\delta)+\delta}\right), \\ &\cong \hat{H} + i\hbar^{\kappa-1}[\hat{P}_0, \hat{H}] + O\left(\hbar^{2\kappa-3\delta}\right). \end{aligned}$$

On the other hand, one can apply Lemma 10 which insures that the symbol of $[\hat{P}_0, \hat{H}]$ is equal to $\frac{\hbar}{i}\{P_0, H\} + O(\hbar^{2-\delta})$. This yields

$$\begin{aligned} e^{i\hbar^{\kappa-1}\hat{P}_0}\hat{H}e^{-i\hbar^{\kappa-1}\hat{P}_0} &\cong \hat{H} + i\hbar^{\kappa-1}\left(\frac{\hbar}{i}\widehat{\{P_0, H\}}\right) + O\left(\hbar^{1+\kappa-\delta}\right) + O\left(\hbar^{2\kappa-3\delta}\right) \\ &\cong \hat{H} + \hbar^\kappa\widehat{\{P_0, H\}} + O\left(\hbar^{\kappa+\alpha}\right), \end{aligned}$$

where we have used the previously defined parameter α .

• Similarly, Lemma 14 tells us that

$$\begin{aligned} e^{i\hbar^{\kappa-1}\hat{P}_0}\hbar^\kappa\hat{K}_0e^{-i\hbar^{\kappa-1}\hat{P}_0} &\cong \hbar^\kappa\hat{K}_0 + O\left(\hbar^\kappa\hbar^{\kappa-1-\delta+1-\delta}\right), \\ &\cong \hbar^\kappa\hat{K}_0 + O\left(\hbar^{\kappa+\alpha}\right), \end{aligned}$$

where we have used that $\kappa - 2\delta > \kappa - 3\delta \geq \alpha$ and thus $\hbar^{\kappa-2\delta} \ll \hbar^\alpha$.

We then write the symbol of Equation (4) and, dividing by \hbar^κ , one sees that we have to solve

$$\{P_0, H\} + K_0 - A_0 = O(\hbar^\alpha). \quad (5)$$

Actually, Lemma 29 insures that we can find a symbol $P'_0 \in \Psi_\delta^{-\delta}$ such that we have exactly $\{P'_0, H\} + K_0 - A'_0 = 0$, with $A'_0(\hbar) \in \Psi_\delta^0$ the \hbar^δ -average of $K_0(\hbar)$. Nevertheless, neither A'_0 nor P'_0 are self-adjoint. The adjoint of this equation is $\{P'_0, H\}^* + K_0 - (A'_0)^* = 0$ since K_0 is self-adjoint. Moreover, using Lemma 9 and Lemma 10, one sees that $\{P'_0, H\}^* = \{(P'_0)^*, H\} + O(\hbar^{1-\delta})$. This implies that $P_0 = \frac{1}{2}(P'_0 + (P'_0)^*)$ satisfies Equation (5) with $A_0 = \frac{1}{2}(A'_0 + (A'_0)^*)$ being the self-adjoint \hbar^δ -average of $K_0(\hbar)$, thanks to Proposition 28 and to the fact that $\hbar^{1-\delta} \leq \hbar^\alpha$. The quantized of P_0 thus satisfies Equation (4) with a self-adjoint $\hat{K}_1 \in \hat{\Psi}_\delta^\alpha$. If we define $\hat{V}_0 = e^{i\hbar^{\kappa-1}\hat{P}_0}$, we have

$$\hat{V}_0(\hat{H} + \hbar^\kappa\hat{K}_0)\hat{V}_0^{-1} \cong \hat{H} + \hbar^\kappa\hat{A}_0 + \hbar^\kappa\hat{K}_1. \quad (6)$$

This equation is the initial step of the following induction process. Let us suppose that at the step $n \geq 0$, we have found self-adjoint PDOs $\hat{K}_1, \dots, \hat{K}_{n+1}$, with $\hat{K}_j \in \hat{\Psi}_\delta^{j\alpha}$ and unitary operators $\hat{V}_0, \dots, \hat{V}_n$ such that

$$\hat{V}_n \dots \hat{V}_0(\hat{H} + \hbar^\kappa\hat{K}_1)\hat{V}_0^{-1} \dots \hat{V}_n^{-1} \cong \hat{H} + \hbar^\kappa \sum_{j=0}^n \hat{A}_j + \hbar^\kappa\hat{K}_{n+1}, \quad (7)$$

where $A_j(\hbar) \in \Psi_\delta^{j\alpha}$ is the self-adjoint \hbar^δ -average of $K_j(\hbar)$. Then we look for a PDO $\hat{K}_{n+2} \in \hat{\Psi}_\delta^{(n+2)\alpha}$ and a unitary operator \hat{V}_{n+1} satisfying

$$\hat{V}_{n+1} \hat{V}_n \dots \hat{V}_0 \left(\hat{H} + \hbar^\kappa \hat{K} \right) \hat{V}_0^{-1} \dots \hat{V}_n^{-1} \hat{V}_{n+1}^{-1} \cong \hat{H} + \hbar^\kappa \sum_{j=0}^{n+1} \hat{A}_j + \hbar^\kappa \hat{K}_{n+2}, \quad (8)$$

where $A_{n+1}(\hbar) \in \Psi_\delta^{(n+1)\alpha}$ is the self-adjoint \hbar^δ -average of $K_{n+1}(\hbar)$. Looking for \hat{V}_{n+1} of the form $\hat{V}_{n+1} = e^{i\hbar^{\kappa-1}\hat{P}_{n+1}}$, with $\hat{P}_{n+1} \in \hat{\Psi}_\delta^{-\delta+(n+1)\alpha}$ self-adjoint, and inserting Equation (7) into Equation (8), we obtain

$$e^{i\hbar^{\kappa-1}\hat{P}_{n+1}} \left[\hat{H} + \hbar^\kappa \sum_{j=0}^n \hat{A}_j + \hbar^\kappa \hat{K}_{n+1} \right] e^{-i\hbar^{\kappa-1}\hat{P}_{n+1}} \cong \hat{H} + \hbar^\kappa \sum_{j=0}^{n+1} \hat{A}_j + \hbar^\kappa \hat{K}_{n+2}. \quad (9)$$

We now apply Lemmas 14 and 15 for each term inside the bracket [].

- First, Lemma 15 tells us that

$$\begin{aligned} \hat{V}_{n+1} \left[\hat{H} \right] \hat{V}_{n+1}^{-1} &\cong \hat{H} + i\hbar^{\kappa-1} \left[\hat{P}_{n+1}, \hat{H} \right] + O \left(\hbar^{2(\kappa-1-\delta+(n+1)\alpha+1-\delta)+\delta} \right) \\ &\cong \hat{H} + i\hbar^{\kappa-1} \left[\hat{P}_{n+1}, \hat{H} \right] + O \left(\hbar^{\kappa+(n+2)\alpha} \right), \end{aligned}$$

where we have used $\hbar^{\kappa-3\delta} \leq \hbar^\alpha$ and $\hbar^{(2n+3)\alpha} \ll \hbar^{(n+2)\alpha}$ provided $n \geq 0$.

- On the other hand, one can apply Lemma 10 which insures that the symbol of $\left[\hat{P}_{n+1}, \hat{H} \right]$ equals to $\frac{\hbar}{i} \{P_{n+1}, H\} + O \left(\hbar^{2-\delta+(n+1)\alpha} \right)$, i.e.

$$\frac{\hbar}{i} \{P_{n+1}, H\} + O \left(\hbar^{1+(n+2)\alpha} \right),$$

where we have used $\hbar^{1-\delta} \leq \hbar^\alpha$. Therefore, the symbol of $\hat{V}_{n+1} \left[\hat{H} \right] \hat{V}_{n+1}^{-1}$ is

$$H + \hbar^\kappa \{P_{n+1}, H\} + O \left(\hbar^{\kappa+(n+2)\alpha} \right).$$

- Then, Lemma 14 provides, for all $j = 0..n$,

$$\begin{aligned} \hat{V}_{n+1} \left[\hbar^\kappa \hat{A}_j \right] \hat{V}_{n+1}^{-1} &\cong \hbar^\kappa \hat{A}_j + O \left(\hbar^\kappa \hbar^{j\alpha} \hbar^{\kappa-1-\delta+(n+1)\alpha+1-\delta} \right) \\ &\cong \hbar^\kappa \hat{A}_j + O \left(\hbar^{\kappa+(n+2)\alpha} \right). \end{aligned}$$

where we have used $\hbar^{\kappa-2\delta} \ll \hbar^{\kappa-3\delta} \leq \hbar^\alpha$ and $\hbar^{j\alpha} \leq 1$.

- Finally, Lemma 14 yields

$$\begin{aligned} \hat{V}_{n+1} \left[\hbar^\kappa \hat{K}_{n+1} \right] \hat{V}_{n+1}^{-1} &\cong \hbar^\kappa \hat{K}_{n+1} + O \left(\hbar^\kappa \hbar^{(n+1)\alpha} \hbar^{\kappa-1-\delta+(n+1)\alpha+1-\delta} \right) \\ &\cong \hbar^\kappa \hat{K}_{n+1} + O \left(\hbar^{\kappa+(2n+3)\alpha} \right) \\ &\cong \hbar^\kappa \hat{K}_{n+1} + O \left(\hbar^{\kappa+(n+2)\alpha} \right), \end{aligned}$$

where we have used $\hbar^{\kappa-2\delta} \ll \hbar^\alpha$ and $\hbar^{(2n+3)\alpha} \ll \hbar^{(n+2)\alpha}$ provided $n \geq 0$.

If we consider these different estimates, if we take the symbol of Equation (9) and if we divide by \hbar^κ , we see that we have to solve

$$\{P_{n+1}, H\} + K_{n+1} - A_{n+1} = O\left(\hbar^{(n+2)\alpha}\right), \quad (10)$$

where $A_{n+1}(\hbar)$ is the self-adjoint \hbar^δ -average of $K_{n+1}(\hbar)$. Then we use Proposition 29 which insures that we can find a symbol $P'_{n+1} \in \Psi_\delta^{-\delta+(n+1)\alpha}$, such that we have exactly $\{P'_{n+1}, H\} + K_{n+1} - A'_{n+1} = 0$, where $A'_{n+1}(\hbar)$ is the \hbar^δ -average of $K_{n+1}(\hbar)$. Using the same technique as for the initial step $n = 0$, we show that $P_{n+1} = \frac{1}{2} \left(P'_{n+1} + \left(P'_{n+1} \right)^* \right)$ satisfies Equation (10) with $A_{n+1} = \frac{1}{2} \left(A'_{n+1} + \left(A'_{n+1} \right)^* \right)$ being the self-adjoint \hbar^δ -average of $K_{n+1}(\hbar)$. The quantized of P_{n+1} thus satisfies Equation (9) with a self-adjoint $\hat{K}_{n+2} \in \hat{\Psi}_\delta^{(n+2)\alpha}$. This concludes the iterative process.

Finally, we apply Borel's construction (Lemma 7) to the sequence of unitary operators $\hat{U}_n = \hat{V}_n \dots \hat{V}_0$ and obtain an unitary operator \hat{U} which satisfies $\left\| \hat{U} - \hat{U}_{n-1} \right\|_{\mathcal{L}(L^2)} = O\left(\hbar^{\kappa-1+\delta+\alpha n}\right)$ for each n . Similarly, Lemma 6 insures that there exists a self-adjoint PDO $\hat{K} \in \hat{\Psi}_\delta^0$ verifying $\hat{K} \sim \sum_n \hat{K}_n$. Moreover, if we define A_\hbar as the self-adjoint \hbar^δ -average of the symbol K_\hbar , we can see that $\hat{A} \sim \sum_n \hat{A}_n$. Therefore, we have proved that the operator \hat{R} defined by

$$\hat{U} \left(\hat{H} + \hbar^\kappa \hat{K}_0 \right) \hat{U}^{-1} = \hat{H} + \hbar^\kappa \hat{A} + \hat{R},$$

satisfies $\left\| \hat{R} \right\|_{\mathcal{L}(L^2)} = O\left(\hbar^\infty\right)$. □

4 Application : quasimodes

As an example of application of the normal form given in Theorem 30, we can easily construct quasimodes associated with the block \mathcal{B}_0 .

Theorem 31 (Non-resonant quasimodes). *Let $\hat{H} \in \hat{\Psi}^0$ be a PDO with non-degenerate CI symbol $H(\xi) \in \Psi^0$ and $\hbar^\kappa \hat{K}_0 \in \hat{\Psi}_\delta^\kappa$ any self-adjoint perturbation with $\kappa > 0$. Let us fix small parameters $\gamma \geq 0$ and $\delta \geq 0$ such that $\delta < 1 - \gamma$ and $\delta < \frac{\kappa}{3}$, and consider the covering of \mathcal{B} with resonance blocks.*

Then for each family $k_\hbar \in \Lambda^$ such that $\hbar k_\hbar$ remains in the non-resonant block \mathcal{B}_0 , there is a \hbar^∞ -quasimode φ_\hbar of the perturbed operator*

$$\left\| \left(\hat{H} + \hbar^\kappa \hat{K}_0 - E_\hbar \right) \varphi_\hbar \right\| = O\left(\hbar^\infty\right)$$

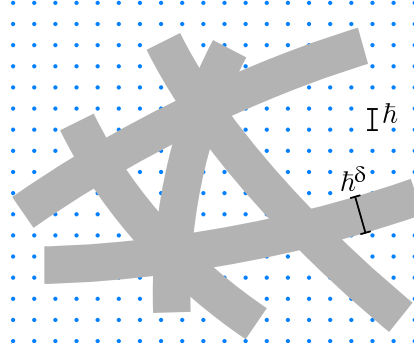
with the property $\varphi_\hbar = e^{ik_\hbar x} + O\left(\hbar^{\kappa-1-\delta}\right)$ and with quasi-eigenvalue

$$E_\hbar = H\left(\hbar k_\hbar\right) + \hbar^\kappa F_\hbar\left(\hbar k_\hbar\right),$$

where the x -independent symbol $F_{\hbar} \in \Psi_{\delta}^0$ is given by

$$F_{\hbar}(\hbar k_{\hbar}) = \langle\langle K_0 \rangle\rangle(\hbar k_{\hbar}) + O(\hbar^{\alpha}),$$

with $\langle\langle K_0 \rangle\rangle(\xi)$ being the average of K_0 over the torus and α being defined by $\alpha = \min(1 - \delta, \kappa - 3\delta)$.



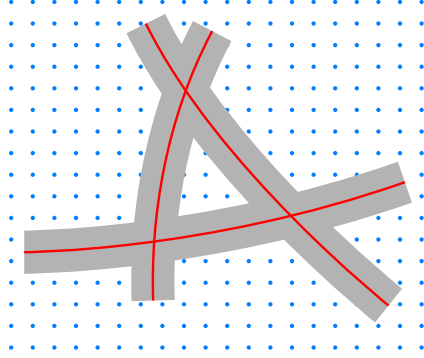
The picture above illustrates the case $d = 2$. The grey region is the zone Z_1^* of 1-resonances and the dots stand for the lattice $\hbar\mathbb{Z}^d$.

Proof. Indeed, according to Theorem 30, the perturbed operator is conjugate to its normal form $\widehat{NF} \cong \hat{H} + \hbar^{\kappa} \hat{A}$, where A_{\hbar} is the self-adjoint \hbar^{δ} -average of K_{\hbar} , which is a symbol satisfying $\hat{K} = \hat{K}_0 + O(\hbar^{\alpha})$. Moreover, in the block \mathcal{B}_0 , Proposition 28 tells us that A_{\hbar} is simply $A_{\hbar} = \langle\langle K_{\hbar} \rangle\rangle + O(\hbar^{\infty})$. On the other hand, as the averaged symbol $\langle\langle K_{\hbar} \rangle\rangle$ is independent on x , the eigenvalues of $\hat{H} + \hbar^{\kappa} \langle\langle K_{\hbar} \rangle\rangle$ are given by $E_k(\hbar) = H(\hbar k) + \hbar^{\kappa} \langle\langle K_{\hbar} \rangle\rangle(\hbar k)$, for each $k \in \Lambda^*$, and the associated eigenvectors are simply the exponential functions e^{ikx} . These functions are thus also \hbar^{∞} -quasimodes of the \widehat{NF} for each family k_{\hbar} such that $\hbar k_{\hbar} \in \mathcal{B}_0$ for all \hbar , and the quasi-eigenvalues are $E_{k_{\hbar}}(\hbar)$. Then, applying the operators \hat{U} which conjugate the perturbed operator to \widehat{NF} , we obtain \hbar^{∞} -quasimodes $\varphi_{\hbar} = \hat{U}^*(e^{ik_{\hbar}x})$ of the perturbed operator, with the same quasi-eigenvalues. Finally, we notice that according to the properties of \hat{U} , the quasimodes have the form $\varphi_{\hbar} = e^{ik_{\hbar}x} + O(\hbar^{\kappa-1-\delta})$. Moreover, according to the expression of K_{\hbar} , the eigenvalues have the expression

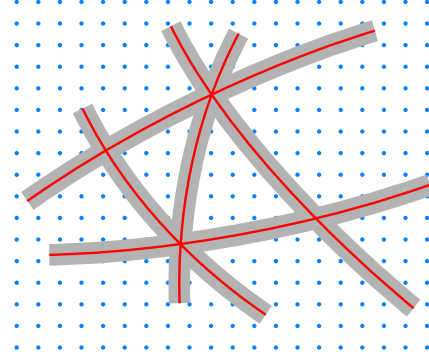
$$E_{k_{\hbar}}(\hbar) = H(\hbar k_{\hbar}) + \hbar^{\kappa} \langle\langle K_0 \rangle\rangle(\hbar k_{\hbar}) + O(\hbar^{\kappa+\alpha}).$$

□

These quasi-eigenvalues are very easily constructed but the number we can construct depends on the size of the block \mathcal{B}_0 as \hbar goes to zero. This size depends on the parameters γ and δ , which control respectively the amount of resonance zones we consider and their width (see below).



"small" δ and γ



"big" δ and γ

With an appropriate choice of γ and δ , one can insure that the relative volume of \mathcal{B}_0 tends to 1 as \hbar goes to zero, as follows.

Proposition 32. *Let $\gamma > 0$ and $\delta > 0$ such that $\delta > 2d\gamma$. For each ball $\mathcal{O} \subset \mathcal{B}$ and each $n = 1..d$, the volume of the block of n -resonances \mathcal{B}_n^* is of order*

$$\text{vol}(\mathcal{B}_n^* \cap \mathcal{O}) = \left(\hbar^{n(\delta-2d\gamma)} \right).$$

Proof. First, one can show (see e.g. [19] or [17]) that the nondegeneracy condition for H implies that there exist two constants T and C such that for all $k \neq 0$ and all ξ satisfying $\frac{\Omega_k(\xi)}{|k|} \leq T$, one has $\left| d\frac{\Omega_k(\xi)}{|k|} \right| \geq C$. For \hbar small enough ($\hbar^{\delta-\gamma n} \leq T$) and for each $k \neq 0$, the points ξ in the resonant zone $\mathcal{Z}_k \cap \mathcal{O}$ satisfy $\left| d\frac{\Omega_k(\xi)}{|k|} \right| \geq C$. The function $\frac{\Omega_k(\xi)}{|k|}$ is thus suitable to measure lengths and this implies that the volume of the resonant zone $\mathcal{Z}_k \cap \mathcal{O}$, for any n -dimensional lattice \mathcal{R} , is bounded by $\text{vol}(\mathcal{Z}_k \cap \mathcal{O}) = O((\hbar^{\delta-\gamma n})^n)$ uniformly with respect to \mathcal{R} . On the other hand, the volume of $\mathcal{B}_k \cap \mathcal{O}$ is bounded in the same way since $\mathcal{B}_k \subset \mathcal{Z}_k$.

The bloc $\mathcal{B}_1^* \cap \mathcal{O}$ of 1-resonances is the union of the $\mathcal{B}_k \cap \mathcal{O}$ for all non-vanishing k with $|k| \leq \hbar^{-\gamma}$. If we bound the sum over all primitive k with $|k| \leq \hbar^{-\gamma}$ by the sum over all k with $|k| \leq \hbar^{-\gamma}$, we get the following estimate

$$\text{vol}(\mathcal{B}_1^* \cap \mathcal{O}) \leq C \hbar^{\delta-\gamma} \sum_{|k| \leq \hbar^{-\gamma}} = O\left(\hbar^{\delta-\gamma-d\gamma}\right).$$

Similarly, for $n = 2..d$, the block $\mathcal{B}_n^* \cap \mathcal{O}$ of n -resonances is the union of the blocks $\mathcal{B}_k \cap \mathcal{O}$ for all the resonance $\hbar^{-\gamma}$ -lattices \mathcal{R} , i.e. the sub-lattices which admit a basis (e_1, \dots, e_n) composed of vector with norm $|e_j| \leq \hbar^{-\gamma}$.

The volume of $\mathcal{B}_n^* \cap \mathcal{O}$ is then roughly bounded by

$$\text{vol}(\mathcal{B}_n^* \cap \mathcal{O}) \leq C' \hbar^{n(\delta-n\gamma)} \sum_{\substack{|e_1| \leq \hbar^{-\gamma} \\ \vdots \\ |e_n| \leq \hbar^{-\gamma}}} = O\left(\hbar^{n(\delta-n\gamma)-\gamma dn}\right).$$

This estimate is actually too rough when $n \geq \frac{d}{2}$ since, for example, the number of lattices of d -resonances is equal to 1 rather than to $\hbar^{-\gamma d^2}$. Therefore, for $n \geq \frac{d}{2}$, we will rather count the orthogonal lattices. This gives a number of lattices of order $\hbar^{-\gamma d(d-n)}$ and thus a volume of order $\text{vol}(\mathcal{B}_n^* \cap \mathcal{O}) = O(\hbar^{n(\delta-n\gamma)-\gamma d(d-n)})$ for $n \geq \frac{d}{2}$. On the other hand, the inequality involving γ and δ implies the following :

- When $0 < n < \frac{d}{2}$, we have

$$n(\delta - n\gamma) - \gamma dn > n\left(\delta - \frac{d}{2}\gamma - \gamma d\right) > n(\delta - 2\gamma d).$$

- When $\frac{d}{2} \leq n \leq d$, we have $n(\delta - n\gamma) - \gamma d(d - n) = n\left(\delta - n\gamma + d\gamma - \gamma \frac{d^2}{n}\right)$ and the bracket is estimated by

$$\delta - n\gamma + d\gamma - \gamma \frac{d^2}{n} \geq \delta - d\gamma + d\gamma - \gamma \frac{2d^2}{d} = \delta - 2\gamma d.$$

This thus shows that the volume $\text{vol}(\mathcal{B}_n^* \cap \mathcal{O})$ is of order $O(\hbar^{n(\delta-2\gamma d)}) \ll 1$ for all $n = 1..d$. \square

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